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Conditions for the Uniqueness of Best Generalized Rational Chebyshev Approximation to Differentiable and Analytic Functions

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1. INTRODUCTION

In the paper [3], we used an optimization theoretical approach to show that the generalized Haar condition is necessary and sufficient for the uniqueness of the best generalized rational Chebyshev approximation to functions defined on a compact Hausdorff space. This general approach includes, in a unified way, weighted, one-sided, asymmetric, and also more general Chebyshev approximation problems with side conditions. It has been known for many years that, in the case of ordinary Chebyshev approximation, best linear or rational approximation to differentiable functions can be unique even when the generalized Haar-condition is not fulfilled. In 1956 Collatz [6] showed that in a strictly convex region of the plane, the linear polynomial of best Chebyshev approximation to a function with continuous first partial derivatives is unique. Four years later Rivlin and Shapiro [12] generalized this result to linear polynomials in several variables and showed that no extension to polynomials of degree higher than one is possible. These results were not derived from a general uniqueness condition for the approximation of differentiable functions and the authors used the special structure of the space of linear polynomials. General uniqueness conditions were given

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by Garkavi [8] and later by Brosowski [1] for the case of linear approximation to differentiable functions defined on a compact interval. These conditions correspond to condition (β) (resp. (α)) in our main theorem. These results were also extended to ordinary rational Chebyshev approximation by Brosowski [1], Brosowski and Loeb [4], and Browoski and Stoer [5]. The extension to manifolds was first considered by Müller [10], but his results do not include the above-mentioned results.

In this paper we use the same optimization theoretical approach of [3] to derive necessary and sufficient conditions for the uniqueness of best rational Chebyshev approximation to differentiable and real analytic functions defined on a compact differentiable manifold (resp. real analytic manifold). As in [3], our results include, besides the ordinary Chebyshev approximation, weighted, one-sided, asymmetric, and also general approximation problems with side conditions. From our general uniqueness conditions we derive the results of Collatz [6] and of Rivlin and Shapiro [12], and also improvements of their results. Further, we show that certain subspaces of quadratic polynomials always satisfy our uniqueness conditions.

It should be mentioned that there exist linear subspaces of $C^k(S)$ of arbitrary high finite dimensions which satisfy the uniqueness condition when S is a compact manifold of dimension 1. However, it is not known whether the same is true when S has dimension $\geqslant 2$.

Now we introduce the necessary definitions. The minimization problem we will consider is:

Let S be a compact n-dimensional real manifold of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, where C^ω denotes the analytic case. The manifold S can be with or without boundary, that is, for each chart (W, φ) the set W is mapped homeomorphically onto an open subset of

$$\mathbb{R}^n_+ := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_1 \geqslant 0\}.$$

Define the compact Hausdorff space $T := \{-1, 1\} \times S$. Let t_0 be any point not in T and let T_0 denote the compact Hausdorff space $T \cup \{t_0\}$ with t_0 as an isolated point.

Let $\{g_1, g_2, \ldots, g_l\}$ and $\{h_1, h_2, \ldots, h_m\}$ be in $C^k(S)$ and, for every $t = (\eta, s) \in T$, define the vectors

$$B(t) := \eta\bar{B}(s) := \eta(g_1(s), g_2(s), \ldots, g_l(s), 0, 0, \ldots, 0),$$

$$C(t) := \ C(s) := (0, 0, \ldots, 0, h_1(s), h_2(s), \ldots, h_m(s)),$$

of $\mathbb{R}^{l+m}$. In the following we will assume that the open convex set

$$U := \bigcap_{t \in T} \{v \in \mathbb{R}^{l+m} \mid \langle C(t), v \rangle > 0\}$$

is nonempty, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^{l+m}$.

Further let $\gamma : T \to \mathbb{R}$ be a nonnegative function such that $\gamma(1, \cdot)$ and $\gamma(-1, \cdot)$ are in $C^k(S)$. For every $(t, v, z) \in T_0 \times U \times \mathbb{R}$ with

$$v = (\alpha_1, \alpha_2, \ldots, \alpha_l, \beta_1, \beta_2, \ldots, \beta_m),$$

we define

$$\begin{aligned} A(t, v, z) &:= z && \text{if} \quad t = t_0, \\ &:= \frac{\langle B(t), v\rangle}{\langle C(t), v\rangle} - \gamma(\eta, s)\, z && \text{if} \quad t \in T. \end{aligned}$$

Then for every x in $C^k(S)$, we consider the minimization problem MPR(x):

$$\text{Minimize} \qquad p(v, z) := z \qquad \text{subject to}$$

$$\underset{(\eta, s) \in T}{\forall} \quad A(\eta, s, v, z) \leqslant \eta x(s).$$

The problem MPR(x) is equivalent to certain rational Chebyshev approximation problems. In fact, consider

$$V := \left\{ \frac{\sum_{i=1}^{l} \alpha_i g_i}{\sum_{i=1}^{m} \beta_i h_i} \in C(S) \;\middle|\; \underset{s \in S}{\forall} \; \sum_{i=1}^{m} \beta_i h_i(s) > 0 \right\}.$$

If $\gamma(\eta, s) = \omega(s) > 0$, then the problem MPR$(x)$ is equivalent to the problem of finding a best rational Chebyshev approximation to x from V with weight function ω, that is, $(v_0, z_0) \in U \times \mathbb{R}$ with

$$v_0 = (\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0l}, \beta_{01}, \beta_{02}, \ldots, \beta_{0m})$$

is a solution of MPR(x) iff

$$z_0 = \left\| \frac{x - r_0}{\omega} \right\|_\infty = \inf_{r \in V} \left\| \frac{x - r}{\omega} \right\|_\infty,$$

where

$$r_0 = \frac{\sum_{i=1}^{l} \alpha_{0i} g_i}{\sum_{i=1}^{m} \beta_{0i} h_i}.$$

If $\gamma(\eta, s) = ((1 + \eta)/2)\, \omega(s)$ resp. $\gamma(\eta, s) = ((1 - \eta)/2)\, \omega(s))$, where ω is a strictly positive continuous function on S, we have one-sided best rational Chebyshev approximation to x from

$$V^+ := \left\{ r \in V \;\middle|\; \underset{s \in S}{\forall} \; r(s) \geqslant x(s) \right\}$$

$$\left(\text{resp. } V^- := \left\{ r \in V \;\middle|\; \underset{s \in S}{\forall} \; r(s) \leqslant x(s) \right\} \right),$$

with weight function ω.

More generally, if $\gamma(1, s) = 0$ (resp. $\gamma(-1, s) = 0$) for some s, we obtain $r(s) \leqslant x(s)$ (resp. $r(s) \geqslant x(s)$).

If $\gamma(1, s) = \gamma(-1, s) = 0$, then $r(s) = x(s)$, that is, the problem MPR includes also best Chebyshev approximation with interpolatory side conditions.

For each $r_0 \in V$ define the linear subspace

$$L(r_0) := \{ \langle B, v \rangle - r_0 \langle C, v \rangle \in C^k(S) \mid v \in \mathbb{R}^{l+m} \}.$$

Let $\{u_1, u_2, \ldots, u_d\}$ be a basis for $L(r_0)$ and define the vectors of $\mathbb{R}^{d+1}$:

$$D(t_0) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad D(t) := D(\eta, s) := \begin{pmatrix} \eta u_1(s) \\ \eta u_2(s) \\ \vdots \\ \eta u_d(s) \\ -\gamma(\eta, s) \end{pmatrix} \quad \text{if } t \in T.$$

We say that a subset $M \subset T$ is *critical for* r_0 iff

$$0 \in \operatorname{con}(\{ D(t) \in \mathbb{R}^{d+1} \mid t \in M \cup \{t_0\} \}).$$

Let $f \in C^1(S)$. A point $s_0 \in S$ will be called a *special zero* of f iff

(1) $f(s_0) = 0$, and

(2) $\operatorname{grad} f(s_0) = 0$, or $s_0 \in \partial S$ and $\dim \partial S = 0$, or $s_0 \in \partial S$, $\dim \partial S \geqslant 1$, and $\operatorname{grad}_{\partial S} f(s_0) = 0$,

where $\operatorname{grad}_{\partial S} f(s_0)$ denotes the gradient with respect to the boundary manifold ∂S. It is easy to see that this definition is independent of the chosen chart (W, φ). When no misunderstanding could arise we denote also in other cases the $\operatorname{grad} f \circ \varphi^{-1}$ by $\operatorname{grad} f$.

The main result of this paper is

THEOREM 1.1. *Let S be an n-dimensional real compact manifold of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\gamma(1, \cdot)$, $\gamma(-1, \cdot) \in C^k(S)$ be such that*

$$\underset{s \in S}{\forall} \; \gamma(-1, s) + \gamma(1, s) > 0.$$

Assume $g_1, g_2, \ldots, g_l$, $h_1, h_2, \ldots, h_m$ *belong to* $C^k(S)$ *and, hence,* $V \subset C^k(S)$. *For each* x *in* $C^k(S)$ *there exists at most one best rational Chebyshev approximation from* V *if and only if for each* $r_0 \in V$ *one of the following equivalent conditions is satisfied*:

(α) *For each critical set* $M \subset T$ *for* r_0 *such that*

$$(\eta, s) \in M \Rightarrow (-\eta, s) \notin M,$$

and for each $f \in L(r_0) \backslash \{0\}$ *there exists a pair* $(\eta, s) \in M$, *such that* s *is not a special zero of* f.

(β) *For* $p = 1, 2, \ldots, d$ *we have*: *each element of a set of linearly independent functions*

$$f_1, f_2, \ldots, f_p \in L(r_0)$$

has at most $(d-p)$ *special zeros in the set*

$$Z_p := \bigcap_{i=1}^{p} \{s \in S \mid f_i(s) = 0\}.$$

(γ) *For each critical set* $M \subset T$ *for* r_0 *such that*

$$(\eta, s) \in M \Rightarrow (-\eta, s) \notin M,$$

and for each $r \in V$, $r \neq r_0$, *there exists a pair* $(\eta, s) \in M$, *such that* s *is not a special zero of* $r - r_0$.

In the case $k = 1$ we assume for the necessity part that $\partial S = \varnothing$ or $\dim S = 1$. We do not know whether the theorem is true for the case $k = 1$ without the restrictions mentioned.

2. UNIQUENESS CONDITIONS

In the case of best rational approximation to continuous functions the Haar condition for the spaces $L(r)$ is equivalent to the uniqueness. In [3] we used implicitly that the Haar condition is equivalent to the following condition:

(α_0) *For each critical set* $M \subset T$ *for* r_0 *such that*

$$(\eta, s) \in M \Rightarrow (-\eta, s) \notin M,$$

and for each $f \in L(r_0) \backslash \{0\}$ *there exists a pair* $(\eta, s) \in M$, *such that* s *is not a zero of* f.

The Haar condition could also have been stated:

(β_0) *For $p = 1, 2, \ldots, d$ we have: p linearly independent functions can have at most $(d-p)$ common zeros in S.*

In the case of differentiable functions we have

PROPOSITION 2.1. *If $L(r_0)$ is contained in $C^1(S)$, then the conditions (α), (β), and (γ) of the theorem are equivalent.*

Proof. $(\alpha) \Rightarrow (\beta)$. Assume there exists a set $f_1, f_2, \ldots, f_p$ of linearly independent functions in $L(r_0)$ and a function f_i, $1 \leqslant i \leqslant p$ (we can assume $i = 1$) such that f_1 has $q := d - p + 1$ special zeros in Z_p, say $s_1, s_2, \ldots, s_q$. Consider the linear equations

$$\sum_{v=1}^{d} \alpha_v u_v(s_\kappa) = 0,$$

$\kappa = 1, 2, \ldots, q$. This system has at least p linearly independent solutions; consequently, the rank ρ of the matrix $(u_v(s_\kappa))$ is less than or equal to $(d-p)$.

Since $q = d - p + 1 > d - p \geqslant \rho$, the vectors

$$\omega_j := \begin{pmatrix} u_1(s_j) \\ u_2(s_j) \\ \vdots \\ u_d(s_j) \end{pmatrix}, \qquad j = 1, 2, \ldots, q,$$

are linearly dependent in $\mathbb{R}^d$. Thus, there exist $\alpha_1, \alpha_2, \ldots, \alpha_q \in \mathbb{R}$, not all zero, such that

$$\sum_{j=1}^{q} \alpha_j \omega_j = 0.$$

Without loss of generality, we can assume $\alpha_1 \neq 0$ and $\gamma(\operatorname{sgn} \alpha_1, s_1) > 0$. Now define

$$\begin{aligned} \eta_j &:= 1 && \text{if } \alpha_j \geqslant 0, \\ &:= -1 && \text{if } \alpha_j < 0. \end{aligned}$$

Then there exist nonnegative $\beta_0, \beta_1, \ldots, \beta_q$ with $\sum_{j=0}^{q} \beta_j = 1$, such that

$$\sum_{j=1}^{q} \beta_j \eta_j \omega_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{q} \beta_j \gamma(\eta_j, s_j) = \beta_0,$$

which imply

$$\sum_{j=0}^{q} \beta_j D(t_j) = 0,$$

where $t_j := (\eta_j, s_j)$, $j = 1, 2, \ldots, q$. Consequently, the set

$$M := \{(\eta_j, s_j) \in T \mid j = 1, 2, \ldots, q\}$$

is critical for r_0, has the property

$$(\eta, s) \in M \Rightarrow (-\eta, s) \notin M,$$

and all the points $s_1, s_2, \ldots, s_q$ are special zeros of f_1, contradicting (α).

$(\beta) \Rightarrow (\gamma)$. Assume there is a critical set $M \subset T$ for r_0 such that $(\eta, s) \in M \Rightarrow (-\eta, s) \notin M$, and a rotational function

$$r = \frac{\langle \bar{B}, v\rangle}{\langle C, v\rangle} \quad \text{in} \quad V \backslash \{r_0\},$$

such that for all $(\eta, s) \in M$ the point s is a special zero of $r - r_0$. We can assume that M is finite, say

$$M = \{(\eta_1, s_1), (\eta_2, s_2), \ldots, (\eta_q, s_q)\}.$$

Obviously, $s_1, s_2, \ldots, s_q$ are also zeros of the function

$$f = \langle \bar{B}, v\rangle - r_0 \langle C, v\rangle,$$

which is an element of $L(r_0) \backslash \{0\}$.

Moreover, we have at the points s_j the equations

$$\begin{aligned}
\operatorname{grad}\left(\frac{\langle \bar{B}, v\rangle}{\langle C, v\rangle} - r_0\right) &= \frac{\langle C, v\rangle \operatorname{grad}\langle \bar{B}, v\rangle - \langle \bar{B}, v\rangle \operatorname{grad}\langle C, v\rangle}{\langle C, v\rangle^2} - \operatorname{grad} r_0 \\
&= \frac{1}{\langle C, v\rangle}\left[\operatorname{grad}\langle \bar{B}, v\rangle - \frac{\langle \bar{B}, v\rangle}{\langle C, v\rangle} \operatorname{grad}\langle C, v\rangle - \langle C, v\rangle \operatorname{grad} r_0\right] \\
&= \frac{1}{\langle C, v\rangle}\left[\operatorname{grad}\langle \bar{B}, v\rangle - r_0 \operatorname{grad}\langle C, v\rangle - \langle C, v\rangle \operatorname{grad} r_0\right] \\
&= \frac{1}{\langle C, v\rangle} \operatorname{grad} f,
\end{aligned}$$

which prove that $s_1, s_2, \ldots, s_q$ are also special zeros of f.

Since M is critical, the linear system

$$\sum_{j=1}^{q} \beta_j u_i(s_j) = 0, \qquad i = 1, 2, \ldots, d,$$

has nontrivial solutions. Then the matrix $(u_i(s_j))$ has rank $\rho \leqslant q-1$. Thus, the transposed linear system

$$\sum_{j=1}^{d} \alpha_j u_j(s_i) = 0, \qquad i = 1, 2, \ldots, q,$$

has at least $p := d - \rho$ linearly independent solutions $f_1, f_2, \ldots, f_p$. We can assume $f_1 = f$. Then

$$\{s_1, s_2, \ldots, s_q\} \subset Z_p,$$

and by (β) it follows that $q \leqslant d - p$. Thus

$$\rho + 1 \leqslant q \leqslant d - p = \rho,$$

which is a contradiction.

$(\gamma) \Rightarrow (\alpha)$. Assume, there is a critical subset $M \subset T$ for

$$r_0 = \frac{\langle \bar{B}, v_0 \rangle}{\langle C, v_0 \rangle},$$

such that $(\eta, s) \in M \Rightarrow (-\eta, s) \notin M$, and a function $f \in L(r_0) \setminus \{0\}$ such that for all $(\eta, s) \in M$ the point s is a special zero of f. We can assume that M is finite, say

$$M = \{(\eta_1, s_1), (\eta_2, s_2), \ldots, (\eta_q, s_q)\}.$$

The function f has a representation

$$f = \langle \bar{B}, v \rangle - r_0 \langle C, v \rangle.$$

We choose $\lambda > 0$ such that

$$v_1 := v_0 + \lambda v \in U.$$

The function

$$r_1 := \frac{\langle \bar{B}, v_1 \rangle}{\langle C, v_1 \rangle}$$

belongs to $V \setminus \{r_0\}$ and the difference $r_1 - r_0$ has $s_1, s_2, \ldots, s_q$ as special zeros, which contradicts (γ). ∎

We conclude this section with some examples.

EXAMPLE 2.2. Let $L(r_0) \subset C^1(S)$ satisfy the Haar condition and let p linearly independent functions $f_1, f_2, \ldots, f_p$ in $L(r_0)$, $1 \leqslant p \leqslant d$, be given. By the condition (β_0), the set Z_p contains at most $(d-p)$ elements, hence, each of the functions $f_1, f_2, \ldots, f_p$ can have at most $(d-p)$ special zeros in Z_p, that is, the condition (β) is fulfilled.

There are linear spaces in $C^1(S)$, which do not satisfy the Haar condition but satisfy the condition (β). A simple example is the linear subspace L in $C^1[-\frac{1}{2}, 1]$ generated by the functions 1 and s^2.

Before we present further examples, we give a characterization of the special zeros in the boundary of n-dimensional compact manifolds in $\mathbb{R}^n$. We have

Lemma 2.3. *Let S be an n-dimensional compact* C^1-manifold in $\mathbb{R}^n$. *A point $s_0 \in \partial S$ is a special zero of a function $f \in C^1(\mathbb{R}^n)$ if and only if the boundary ∂S and the set*

$$\Gamma := \{y \in \mathbb{R}^n \mid f(y) = 0\}$$

have a contact of order one in s_0, that is, they have in s_0 the same tangential plane.

Proof. We show that $\operatorname{grad} f(s_0)$ is orthogonal to the tangential plane of ∂S in the point s_0. To determine the tangential plane of ∂S in s_0, choose a chart (W, φ) in ∂S such that $\varphi(s_0) := x_0 \in \mathbb{R}^{n-1}$. Then the tangential plane is given by

$$\operatorname{span}\left(\frac{\partial\varphi^{-1}(x_0)}{\partial x_1}, \frac{\partial\varphi^{-1}(x_0)}{\partial x_2}, \ldots, \frac{\partial\varphi^{-1}(x_0)}{\partial x_{n-1}}\right),$$

where $\partial\varphi^{-1}(x_0)/\partial x_\nu$ denotes the vector

$$\left(\frac{\partial\varphi_1^{-1}(x_0)}{\partial x_\nu}, \frac{\partial\varphi_2^{-1}(x_0)}{\partial x_\nu}, \ldots, \frac{\partial\varphi_n^{-1}(x_0)}{\partial x_\nu}\right), \qquad \nu = 1, 2, \ldots, n-1.$$

By its definition, a point $s_0 \in \partial S$ is a special zero of $f \in C^1(S)$ iff

$$\frac{\partial f \circ \varphi^{-1}(x_0)}{\partial x_\nu} = 0, \qquad \nu = 1, 2, \ldots, n-1.$$

Using the chain rule, the last equations are equivalent to the equations

$$\left\langle \operatorname{grad} f(s_0), \frac{\partial\varphi^{-1}(x_0)}{\partial x_\nu} \right\rangle = 0, \qquad \nu = 1, 2, \ldots, n-1,$$

that is, equivalent to Γ and ∂S have the same tangential plane in s_0. ∎

Example 2.4. Let S be an n-dimensional compact C^1-manifold in $\mathbb{R}^n$ with the following property: If a hyperplane touches the boundary in $q \geqslant 2$

points, then these points are not contained in a $(q-2)$-dimensional plane of $\mathbb{R}^n$. We call such a manifold admissible. Examples of such manifolds are strictly convex C^1-manifolds and the union of two disjoint strictly convex C^1-manifolds.

Then the linear space L of all linear polynomials

$$a_0 + a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$$

satisfies the condition (β).

The space L has dimension $d := n+1$. Choose $p \leqslant n+1$ linearly independent linear polynomials $f_1, f_2, \ldots, f_p$. Since the set Z_{n+1} is empty, we have only to consider the case $p \leqslant n$. Assume there exists a function f_i, $1 \leqslant i \leqslant p$ (we can assume $i = 1$), such that f_1 has $q := d - p + 1 = n - p + 2$ special zeros in Z_p, say $s_1, s_2, \ldots, s_q$. It is easy to see, that these zeros are in ∂S. By Lemma 2.3, the hyperplane

$$H_1 := \{x \in \mathbb{R}^n \mid f_1(x) = 0\}$$

touches ∂S in the points $s_1, s_2, \ldots, s_q$. On the other hand, the $(n-p)$-dimensional set Z_p contains the point $s_1, s_2, \ldots, s_q$. This is impossible, since S is an admissible manifold.

EXAMPLE 2.5. The preceding result does not extend to arbitrary quadratic or higher degree polynomials as the following example shows. Let S be an n-dimensional compact C^1-manifold in $\mathbb{R}^n$ and let

$$f(y) = a_0 + \sum_{\nu=1}^{n} a_\nu y_\nu$$

be a linear polynomial such that the hyperplane

$$\Gamma_f := \{y \in \mathbb{R}^n \mid f(y) = 0\}$$

has a nonempty intersection with the interior of S. Then, the quadratic polynomial f^2 has infinitely many special zeros in S.

However, linear subspaces of quadratic polynomials can satisfy the condition (β) for special manifolds. In fact, let L be the space of all polynomials

$$f(y_1, y_2) = a_0 + a_1 y_1 + a_2 y_2 + a_3(y_1^2 + y_2^2),$$

and let S_E be the C^1-manifold

$$S_E := \left\{ y \in \mathbb{R}^2 \,\middle|\, \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} \leqslant 1 \right\}, \qquad a \neq b.$$

We show that each polynomial $f \neq 0$ has at most two special zeros in S. If $a_3 = 0$, then, by Example 2.4, f has at most one special zero in ∂S_E (strict convexity of S_E). If $a_3 \neq 0$ and y_0 is a special zero of f in the interior of S_E, then f has at the point y_0 its unique maximum or minimum. Thus, f cannot be zero in any other point of S_E. If y_0 is a special zero in the boundary, then Γ_f and ∂S_E have a first-order contact at y_0. Since Γ_f is a circle, it can touch ∂S_E in at most two points.

Next we show that L satisfies condition (β). We have only to consider the cases of three and four linearly independent functions in L. It is easy to see that Z_3 consists of at most one point and that Z_4 is empty.

It should be mentioned that strict convexity of S is not sufficient for L to satisfy condition (β) on S. For instance, let S be the unit circle in $\mathbb{R}^2$. Then the polynomial

$$f(y_1, y_2) = -1 + y_1^2 + y_2^2$$

has all boundary points of S as special zeros.

A further example of a linear subspace of quadratic polynomials which satisfies condition (β) in S_E is given by the polynomials

$$f(y_1, y_2) = a_0 + a_1 y_1 + a_2 y_2 + a_3(y_1^2 - y_2^2) + a_4 y_1 y_2.$$

In this case each $f \neq 0$ can have at most three special zeros (one in int S_E and two in ∂S_E). To prove condition (β) one has to check only the cases of 5, 4, and 3 linearly independent functions. Like before, we can show that $Z_5 = \varnothing$, $\#(Z_4) \leqslant 1$, and $\#(Z_3) \leqslant 2$.

3. THE CONDITIONS ARE SUFFICIENT

The sufficiency part of the theorem follows from Proposition 2.1 and the more general

THEOREM 3.1. *Let S be an n-dimensional compact real manifold of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\gamma(1, \cdot)$, $\gamma(-1, \cdot)$ in $C^k(S)$ be such that*

$$\forall_{s \in S} \; \gamma(1, s) + \gamma(-1, s) > 0.$$

Assume $g_1, g_2, \ldots, g_l$, $h_1, h_2, \ldots, h_m$, belong to $C^k(S)$ and, hence, $V \subset C^k(S)$. Let x be in $C^k(S)$ such that $u_0 := (v_0, z_0)$ and $u_1 := (v_1, z_0)$ are minimal points of MPR(x). *If r_0 satisfies condition (γ), then $r_0 = r_1$.*

Proof. We can assume $x \notin V$. By Theorem 3.1 and Lemma 4.2 of [3], the set

$$M := \{(\eta, s) \in T \mid \eta r_0(s) - \gamma(\eta, s)\, z_0 = \eta r_1(s) - \gamma(\eta, s)\, z_0 = \eta x(s)\}$$

is critical for r_0. Moreover, $(\eta, s) \in M$ and $(-\eta, s) \in M$ would imply

$$\gamma(\eta, s)\, z_0 + \gamma(-\eta, s)\, z_0 = 0.$$

Since $x \notin V$, we have $z_0 \neq 0$ and, hence, $\gamma(\eta, s) + \gamma(-\eta, s) = 0$, which is impossible.

Next we show that for each $(\eta, s) \in M$ the point s is a special zero for $r_1 - r_0$. By the definition of M we have $r_1(s) - r_0(s) = 0$ for every $(\eta, s) \in M$. Moreover, each of the functions

$$\Delta_0 := \eta r_0 - \gamma(\eta, \cdot)\, z_0 - \eta x$$

and

$$\Delta_1 := \eta r_1 - \gamma(\eta, \cdot)\, z_0 - \eta x$$

has a maximum in $s \in W$, for every $(\eta, s) \in M$. Choose a chart (W, φ) such that $s \in W$ and let $y := \varphi(s)$. Then we have

$$\begin{aligned}\frac{\partial \Delta_i \circ \varphi^{-1}}{\partial y_\nu}(y) = \eta \frac{\partial r_i \circ \varphi^{-1}}{\partial y_\nu}(y) - \frac{\partial \gamma \circ \varphi^{-1}}{\partial y_\nu}(y) \\ - \frac{\partial x \circ \varphi^{-1}}{\partial y_\nu}(y) = 0,\end{aligned}$$

for $i = 0, 1$ and $\nu = 1, 2, \ldots, n$, if s is an interior point of S and $\nu = 2, 3, \ldots, n$, if $s \in \partial S$. These equations imply

$$\begin{aligned}&\frac{\partial \Delta_0 \circ \varphi^{-1}}{\partial y_\nu}(y) - \frac{\partial \Delta_1 \circ \varphi^{-1}}{\partial y_\nu}(y) \\ &\quad = \eta \left(\frac{\partial r_0 \circ \varphi^{-1}}{\partial y_\nu}(y) - \frac{\partial r_1 \circ \varphi^{-1}}{\partial y_\nu}(y)\right) = 0,\end{aligned}$$

for $\nu = 1, 2, \ldots, n$, if $s \in \operatorname{int}(S)$ and $\nu = 2, 3, \ldots, n$, if $s \in \partial S$. Hence, s is a special zero of $r_0 - r_1$. By condition (β) we have $r_0 = r_1$. ∎

EXAMPLE 3.2. Let S be the unit circle in $\mathbb{R}^2$ and let L be the linear space of all linear polynomials

$$a_0 + a_1 y_1 + a_2 y_2.$$

By Example 2.4, L satisfies condition (β) on S. Consequently, there exists a unique linear polynomial of best approximation to each $x \in C^1(S)$.

In the case of ordinary best Chebyshev approximation this result is due to Collatz [6], who proved it in a different way. The result we present here is more general, since it includes also other types of Chebyshev approximation like, for instance, one-sided and asymmetric approximation.

EXAMPLE 3.3. Let S be an n-dimensional compact C^1-manifold in $\mathbb{R}^n$ which is admissible in the sense of Example 2.4. Let L be the linear space of all linear polynomials

$$a_0 + \sum_{\nu=1}^{n} a_\nu y_\nu.$$

By Example 2.4, L satisfies condition (β) on S. So, there exists a unique linear polynomial of best approximation to each $x \in C^1(S)$.

In the case $\gamma(\eta, s) = 1$ (ordinary Chebyshev approximation), this result is essentially due to Rivlin and Shapiro [12], who proved it in a different way. Like in the example before, our result includes other types of approximation problems.

EXAMPLE 3.4. Let S_E be the C^1-manifold

$$\left\{ y \in \mathbb{R}^2 \;\middle|\; \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} \leqslant 1 \right\}, \qquad a \neq b,$$

and let L_1 (resp. L_2), denote the linear space of all polynomials

$$a_0 + a_1 y_1 + a_2 y_2 + a_3(y_1^2 + y_2^2)$$

(resp. $a_0 + a_1 y_1 + a_2 y_2 + a_3(y_1^2 - y_2^2) + a_4 y_1 y_2$).

By Example 2.5, the spaces L_1 and L_2 satisfy condition (β) on S_E. So, there exists for each $x \in C^1(S_E)$ a unique best approximation from L_1 (resp. from L_2).

4. THE CONDITIONS ARE NECESSARY

The necessity part of the theorem follows from Proposition 2.1 and the more general

THEOREM 4.1. *Let S be an n-dimensional compact real manifold of class* C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, *and let* $\gamma(1, \cdot)$, $\gamma(-1, \cdot)$ *in* $C^k(S)$ *be such that*

$$\underset{s \in S}{\forall}\ \gamma(1, s) + \gamma(-1, s) > 0.$$

Assume $g_1, g_2, \ldots, g_l$, $h_1, h_2, \ldots, h_m$, *belong to* $C^k(S)$ *and, hence,* $V \subset C^k(S)$. *In the case* $k = 1$, *we assume that* $\partial S = \emptyset$ *or* $\dim S = 1$. *If there is an* $r_0 \in V$ *which does not satisfy condition* (γ), *then we can find a function* $x \in C^k(S)$ *such that the problem* MPR(x) *has two minimal points* (v_0, z_0) *and* (v_1, z_0) *with* $r_0 - r_1 \neq 0$.

The proof of this theorem is an immediate consequence of the next lemmas. For the proof of the lemmas, we remark that there exist functions

$$x_\mu := S \to \mathbb{R}, \qquad \mu = 1, 2, \ldots, q,$$

of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, such that the function

$$S \ni s \to (x_1(s), x_2(s), \ldots, x_q(s)) \in \mathbb{R}^q$$

is of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, injective and with Jacobians of rank n (compare Hirsch [9]).

LEMMA 4.2. *Let* S *be an* n*-dimensional real compact manifold of class* C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$. *In the case* $k = 1$, *we assume that* $\partial S = \emptyset$ *or* $\dim S = 1$. *Let* $N \subset S$ *be a finite set and* ψ_1, $\psi_2 \in C^k(S)$ *be such that each point of* N *is a special zero of* $\psi := \psi_1 - \psi_2$. *Then there exists* $h \in C^k(S)$ *such that*

$$\underset{s \in S}{\forall}\ h(s) \geqslant \max\{\psi_1(s), \psi_2(s)\},$$

and each point of N *is a special zero of* $H - \psi_1$ *and* $H - \psi_2$.

For the proof see Section 5.

LEMMA 4.3. *Let* S *be an* n*-dimensional real compact manifold of class* C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$ *and let* $\gamma(1, \cdot)$, $\gamma(-1, \cdot)$ *in* $C^k(S)$ *be such that*

$$\underset{s \in S}{\forall}\ \gamma(1, s) + \gamma(-1, s) > 0.$$

In the case $k = 1$, *we assume that* $\partial S = \emptyset$ *or* $\dim S = 1$. *Assume further,* $g_1, g_2, \ldots, g_l$, $h_1, h_2, \ldots, h_m$, *belong to* $C^k(S)$ *and, hence,* $V \subset C^k(S)$. *Let* v_0, $v \in U$ *and* $M \subset T$ *be finite such that for all* $(\eta, s) \in M$ *the point* s *is a special zero of* $r_0 - r$. *If* M *is critical for* r_0 *and*

$$(\eta, s) \in M \Rightarrow (-\eta, s) \notin M,$$

then there are $x \in C^k(S)$ *and* $z_0 \in \mathbb{R}$ *such that* (v_0, z_0) *and* (v, z_0) *are minimal points for* MPR(x).

Proof. We first show: If $N \subset T$ is finite, $(\eta, s) \in N \Rightarrow (-\eta, s) \notin N$, and for each $(\eta, s) \in N$ the point s is a special zero of $r_0 - r$, then there are $x \in C^k(S)$ and $z_0 \in \mathbb{R}$ such that

$$\forall_{(\eta,s)\in T} \quad A(\eta, s, v_0, z_0) \leqslant \eta x(s),$$

$$A(\eta, s, v, z_0) \leqslant \eta x(s),$$

and

$$\forall_{(\eta,s)\in N} \quad A(\eta, s, v_0, z_0) = \eta x(s).$$

By Lemma 4.2, there exist functions h, f in $C^k(S)$ such that

$$h(s) \geqslant \max[r_0(s), r(s)],$$

$$f(s) \leqslant \min[r_0(s), r(s)],$$

and for all $(\eta, s) \in N$, the point s is a special zero of

$$h - r_0, \qquad h - r, \qquad f - r_0, \qquad f - r.$$

Further we define

$$z_0 := \max_{s\in S} \frac{h(s) - f(s)}{\gamma(1, s) + \gamma(-1, s)} \geqslant 0.$$

Let $N^+ := \{s \in S \mid (1, s) \in N\}$ and $N^- := \{s \in S \mid (-1, s) \in N\}$. We claim that there exists a fuction g in $C^k(S)$ such that $0 \leqslant g \leqslant 1$, $g \equiv 1$ in N^+, and $g \equiv 0$ in N^-.

For the proof we consider a function $S \ni s \mapsto (x_1(s), x_2(s), ..., x_q(s)) \in \mathbb{R}^q$ of class C^k and injective. For constructing the function g, we define for each point $s_{\kappa_0} \in N$ the function

$$D_{\kappa_0}(s) := \sum_{\mu=1}^{q} (x(s) - x(s_{\kappa_0}))^2.$$

For each point $s_{\kappa_0} \in N^-$, lchoose a real number $\alpha_{\kappa_0} > 0$ so small that

$$\forall_{s\in S} \; 1 - \alpha_{\kappa_0} D_{\kappa_0}(s) > 0,$$

and define the function

$$\Gamma_{\kappa_0}(s) := (1 - \alpha_{\kappa_0} D_{\kappa_0}(s)) \cdot \frac{\prod_{s_\kappa \neq s_{\kappa_0}} D_\kappa(s)}{\prod_{s_\kappa \neq s_{\kappa_0}} D_\kappa(s_{\kappa_0})}.$$

Then the function

$$\Gamma(s) := \Big[1 - \sum_{s_{\kappa_0} \in N^-} \Gamma_{\kappa_0}(s)\Big]^2$$

has the following properties:

(i) $\forall_{s \in N^-}\, \Gamma(s) = 0$;

(ii) each point $s \in N^+$ is a strict local maximum of Γ and we have $\Gamma(s) = 1$ for $s \in N^+$.

By (ii), there exists for each point $s_0 \in N^+$ a neighborhood U_{s_0} of s_0 such that

$$\underset{s \in U_{s_0}}{\forall} \Gamma(s) \leqslant 1.$$

Now consider the set

$$S_1 := \Big\{ s \in S \Big\backslash \bigcup_{s_0 \in N^+} U_{s_0} \,\Big|\, \Gamma(s) > 1 \Big\}.$$

If this set is empty, then we set $g := \Gamma$.

If not, then we can choose an $\eta > 0$ such that the function

$$Q(s) := \Big(1 - \eta \prod_{s_\kappa \in N^+} D_\kappa(s)\Big)$$

is strictly positive in S. Since

$$\sup_{s \in S_1} Q(s) < 1,$$

the function

$$g(s) := \Gamma(s) \cdot Q(s)^p,$$

for p large enough, satisfies $0 \leqslant g \leqslant 1$, $g \equiv 1$ in N^+ and $g \equiv 0$ in N^-. Then the function

$$x(s) := g(s)[h(s) - \gamma(1, s)\, z_0] + (1 - g(s))[f(s) + \gamma(-1, s)\, z_0]$$

has the required properties.

For the other part of the proof, we can use the proof of [3, Lemma 5.3, pp. 162–169] together with [3, Lemma 4.2]. ▮

A corollary of Theorem 4.1 is the following refinement of Theorem 3 of Rivlin and Shapiro [12].

COROLLARY 4.4. *Let S be an n-dimensional compact manifold in $\mathbb{R}^n$ of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$ and let $\gamma(1, \cdot)$, $\gamma(-1, \cdot)$ be in $C^k(S)$ such that*

$$\forall_{s \in S} \ \gamma(-1, s) + \gamma(1, s) > 0.$$

In the case $k = 1$, we assume that $\partial S = \emptyset$ or $\dim s = 1$. *Assume further there exists a hyperplane tangent to the boundary of S at $p \geqslant 3$ distinct points, which are contained in a $(p-2)$-dimensional plane of $\mathbb{R}^n$.*

Then there exists a function $x \in C^k(S)$, which has two best approximations from the linear space L of linear polynomials

$$a_0 + \sum_{\nu=1}^{n} a_\nu y_\nu.$$

If $\gamma(-1, \cdot)$ and $\gamma(1, \cdot)$ are constant functions and $k \neq 1$, then the function x can be chosen as the restriction of a real analytic function defined on $\mathbb{R}^n$.

Proof. We denote by Γ_f the hyperplane $\{y \in \mathbb{R}^n \,|\, f(y) = 0\}$ tangent to the boundary of S in the points of set

$$M_0 := \{s_1, s_2, \ldots, s_p\},$$

where $f(y) := a_0 + \sum_{\nu=1}^{n} a_\nu y_\nu$.

By a theorem of Radon [7], the set M_0 is the union of two disjoint subsets M_1 and M_{-1} of M_0 such that the convex hull of M_1 and the convex hull of M_{-1} have a nonempty intersection. Then the set

$$M := \{(\eta, s) \in T \,|\, s \in M_i \Rightarrow \eta = i\}$$

is critical for the linear space L. By Lemma 2.3, each point of M_0 is a special zero of the linear polynomial f, that is, f has special zeros for each s with (η, s) in the critical set M. Then, by Theorem 4.1 and Proposition 2.1, there exists a function $x \in C^k(S)$ such that x has two best approximations from L.

Now assume $\gamma(-1, 0)$ and $\gamma(1, 0)$ are constant and $k \neq 1$. In the proofs of Lemma 4.2 and 4.3 we used for the construction of the above-mentioned function x a mapping

$$S \ni s \to (x_1(s), x_2(s), \ldots, x_q(s)) \in \mathbb{R}^q. \tag{1}$$

The constructed function x was then a polynomial in the functions $x_1, x_2, \ldots, x_q$, and f. In the case of an n-dimensional manifold in $\mathbb{R}^n$, we can choose $q = n$ and for the mapping (1) the identity map on S. Then the construction leads to a polynomial in the coordinates $y_1, y_2, \ldots, y_n$ which is defined on $\mathbb{R}^n$ and is, of course, analytic. ∎

Remark. Rivlin and Shapiro [12] proved in the case $k=2$, $\gamma(\eta, s)=1$, and $p=n+1$, the existence of a C^∞-function x defined on $\mathbb{R}^n$.

5. PROOF OF LEMMA 4.2

For the case $k=1$ we prove the lemma in a more general situation. We will assume that one of the following conditions is satisfied:

(1) $\partial S=\varnothing$;
(2) $\dim S=1$;
(3) $\partial S \cap N=\varnothing$;
(4) For each $s \in N$ we have $\operatorname{grad} \psi(s)=0$.

These conditions can be used to generalize Theorem 4.1.

It suffices to prove the existence of a function P in $C^k(S)$ such that all points in N are special zeros of P and

$$\underset{s \in S}{\forall} \; P(s) \geqslant \max\{\psi(s), 0\}.$$

Then $H := P + \psi_2$ has the required properties.

As in the remark after Theorem 4.1, consider a function

$$S \ni s \to (x_1(s), x_2(s), \ldots, x_q(s)) \in \mathbb{R}^q$$

of class C^k, injective, and with Jacobian of rank n. For constructing the function P, we define for each point s_{κ_0} in N the function

$$D_{\kappa_0}(s) := \sum_{\mu=1}^{q} (x_\mu(s) - x_\mu(s_{\kappa_0}))^2,$$

and the function

$$J_{\kappa_0}(s) := \sqrt{\psi(s)^2 + a_{\kappa_0}^2 D_{\kappa_0}(s)^2} \qquad \text{if} \quad k=1,$$
$$:= \alpha_{\kappa_0}\psi(s) + a_{\kappa_0} D_\kappa(s) \qquad \text{if} \quad k \neq 1,$$

where α_{κ_0} (resp. a_{κ_0}), are real (resp. positive real) parameters, which will be chosen later.

The functions J_{κ_0} are of class C^k. Only the case $k=1$ needs a proof for the points $s \in S$, where $\psi(s)^2 + a_{\kappa_0}^2 D_{\kappa_0}(s)=0$, that is, for $s=s_{\kappa_0}$. First let s_{κ_0} be a point in $S \backslash \partial S$ and let (W, φ) be a chart with $s_{\kappa_0} \in W$ and $y_0 := \varphi(s_{\kappa_0})$. Then we have

$$\frac{\partial J_{\kappa_0} \circ \varphi^{-1}}{\partial y_\nu}(y_0)=0, \qquad \nu=1, 2, \ldots, n,$$

and for $y \neq y_0$, the estimate

$$\begin{aligned}
&\left|\frac{\partial J_{\kappa_0} \circ \varphi^{-1}}{\partial y_\nu}(y)\right| \\
&\quad = \left|\left(\psi(s)\frac{\partial \psi \circ \varphi^{-1}}{\partial y_\nu}(y) + a_{\kappa_0}^2 D_{\kappa_0}(s)\frac{\partial D_{\kappa_0} \circ \varphi^{-1}}{\partial y_\nu}(y)\right)\Big/\sqrt{\psi(s)^2 + a_{\kappa_0}^2 D_{\kappa_0}^2(s)}\right| \\
&\quad \leqslant -\sqrt{((\partial \psi \circ \varphi^{-1}/\partial y_\nu)(y))^2 + a_{\kappa_0}^2((\partial D_{\kappa_0} \circ \varphi^{-1}/\partial y_\nu)(y))^2},
\end{aligned}$$

$\nu = 1, 2,\dots, n$, which is an application of the Cauchy–Schwartz inequality. This estimate shows that

$$\frac{\partial J_{\kappa_0} \circ \varphi^{-1}}{\partial y_\nu}(y) \to 0 \qquad \text{for} \quad y \to y_0,$$

which proves the continuity of grad J_{κ_0} in $s_{\kappa_0} \in S \backslash \partial S$.

Now let s_{κ_0} be in ∂S and let (W, φ) be a chart such that $s_{\kappa_0} \in W$ and $\varphi(s_{\kappa_0}) = 0$. Then the function $J_{\kappa_0} \circ \varphi^{-1}$ is in a neighborhood of zero the restriction of the continuous function Γ, which is defined by

$$\begin{aligned}
\Gamma(y_1, y_2,\dots, y_n) &:= J_{\kappa_0} \circ \varphi^{-1}(y_1, y_2,\dots, y_n) && \text{if} \quad y_1 > 0, \\
&:= 2J_{\kappa_0} \circ \varphi^{-1}(0, y_2,\dots, y_n) && \\
&:= -J_{\kappa_0} \circ \varphi^{-1}(-y_1, y_2,\dots, y_n) && \text{if} \quad y_1 < 0.
\end{aligned}$$

This function is of class C^1. This is obvious for the points $y \neq 0$. To prove the continuity of grad $\Gamma(0)$ in $y = 0$, we observe that

$$\operatorname{grad} \Gamma(0) = \left(\left|\frac{\partial \psi \circ \varphi^{-1}}{\partial y_1}(0)\right|, 0,\dots, 0\right).$$

Then, we define for each $y \in \mathbb{R}^n$ elements $\tilde{y}$ and $\hat{y}$ by setting

$$\tilde{y} := (0, y_2, y_3,\dots, y_n)$$

and

$$\begin{aligned}
\hat{y} &:= (-y_1, y_2,\dots, y_n) && \text{if} \quad y_1 < 0, \\
&:= y && \text{if} \quad y_1 \geqslant 0.
\end{aligned}$$

Like in the case $s_{\kappa_0} \in S \backslash \partial S$, the use of the Cauchy–Schwartz inequality yields the estimate

$$\begin{aligned}
\left|\frac{\partial \Gamma(y)}{\partial y_\nu}\right| &\leqslant 2\sqrt{((\partial \psi \circ \varphi^{-1}/\partial y_\nu)(\tilde{y}))^2 + a_{\kappa_0}^2((\partial D_{\kappa_0} \circ \varphi^{-1}/\partial y_\nu)(\tilde{y}))^2} \\
&\quad + \sqrt{((\partial \psi \circ \varphi^{-1}/\partial y_\nu)(\hat{y}))^2 + a_{\kappa_0}^2((\partial D_{\kappa_0} \circ \varphi^{-1}/\partial y_\nu)(\hat{y}))^2},
\end{aligned}$$

which proves the continuity of $\partial \Gamma/\partial y_\nu$ in the case $\partial \Gamma(0)/\partial y_\nu = 0$.

Consequently, J_{κ_0} is a C^1-function in the case $k=1$, provided one of the conditions (1), (3), or (4) is satisfied. To prove the case of $\dim S = 1$, we have only to consider $(\partial\Gamma/\partial y_1)(0) \neq 0$. Then we have

$$\frac{\partial\Gamma(y)}{\partial y_1} = \frac{\left[\begin{array}{l}\psi\circ\varphi^{-1}(\hat{y})\cdot(\partial\psi\circ\varphi^{-1}/\partial y_1)(\hat{y}) \\ \qquad + a_{\kappa_0}^2 D_{\kappa_0}\circ\varphi^{-1}(\hat{y})(\partial D_{\kappa_0}\circ\varphi^{-1}/\partial y_1)(\hat{y})\end{array}\right]}{\sqrt{(\psi\circ\varphi^{-1}(\hat{y}))^2 + a_{\kappa_0}^2(D_{\kappa_0}\circ\varphi^{-1}(\hat{y}))^2}}.$$

Dividing on the right-hand side numerator and denominator by y_1, we obtain

$$\frac{\partial\Gamma(y)}{\partial y_1} \to \left|\frac{\partial\psi\circ\varphi^{-1}}{\partial y_1}(0)\right|,$$

for $y\to 0$, which proves the continuity of $\partial\Gamma/\partial y_1$ also in the case $\partial\Gamma(0)/\partial y_1 \neq 0$.

Consequently, J_{κ_0} is also a C^1-function in this case.

The α_{κ_0} are chosen as follows: let (W,φ) be a chart with $s_{\kappa_0}\in W$ such that $y_0=\varphi(s_{\kappa_0})$. Then we set

$$\alpha_{\kappa_0} := \operatorname{sgn}\frac{\partial\psi\circ\varphi^{-1}}{\partial y_1}(y_0).$$

We claim that we can choose real numbers a_{κ_0} so large that

$$J_{\kappa_0}(s) > 0 \qquad \text{if} \quad s \neq s_{\kappa_0}, \tag{2}$$

and that there exists a neighborhood U_{κ_0} of s_{κ_0} such that

$$\underset{s\in U_{\kappa_0}}{\forall}\; J_{\kappa_0}(s) \geqslant \psi(s). \tag{3}$$

Remark. Once we have proved this, then we can also choose the a_κ so large, that for a suitable neighborhood U_{κ_0} (we can assume that this is the same neighborhood as in (3)), we have

$$\underset{\kappa\neq\kappa_0}{\forall}\;\underset{s\in U_{\kappa_0}}{\forall}\; J_\kappa(s)\geqslant 1.$$

The claims (2) and (3) are trivial for $k=1$ and the claim (2) is also for $k\neq 1$ and $\operatorname{grad} J_{\kappa_0}(s_{\kappa_0})=0$.

Assume now $k\neq 1$ and consider for the real numbers $\beta=\alpha_{\kappa_0}$ (resp. $\beta=\alpha_{\kappa_0}-1$) the matrix

$$C_0(\beta) := \left(2a_{\kappa_0}\sum_{\mu=1}^{q}\frac{\partial x_\mu\circ\varphi^{-1}}{\partial y_\nu}(y_0)\cdot\frac{\partial x_\mu\circ\varphi^{-1}}{\partial y_\sigma}(y_0)\right.$$
$$\left.+\beta\frac{\partial^2\psi\circ\varphi^{-1}}{\partial y_\nu\,\partial y_\sigma}(y_0)\right),\qquad \nu=1,2,\ldots,n,\quad \sigma=1,2,\ldots,n.$$

The matrix

$$B_0 := \left(\sum_{\mu=1}^{q} \frac{\partial x_\mu \circ \varphi^{-1}}{\partial y_\nu}(y_0) \cdot \frac{\partial x_\mu \circ \varphi^{-1}}{\partial y_0}(y_0)\right),$$

$$\nu = 1, 2, \ldots, n, \quad \sigma = 1, 2, \ldots, n,$$

is positive definite, since B_0 can be written as a product $A_0 . A_0^{\mathrm{T}}$, where A_0 is the Jacobian matrix

$$\left(\frac{\partial x_\mu \circ \varphi^{-1}}{\partial y_\nu}(y_0)\right), \qquad \nu = 1, 2, \ldots, n, \quad \mu = 1, 2, \ldots, q,$$

which has rank n. We can choose a_{κ_0} so large, that the matrix C_0 (β) is positive definite for both values of β. To prove the claims (2) and (3) in the case $\operatorname{grad} J_{\kappa_0}(s_{\kappa_0}) \neq 0$, we consider the second-order Taylor expansion of J_{κ_0} (resp. $J_{\kappa_0} - \psi$) at the point s_{κ_0}:

$$J_{\kappa_0} \circ \varphi^{-1}(y) = \left|\frac{\partial \psi \circ \varphi^{-1}}{\partial y_1}(0)\right| y_1 + \langle y, C_0(\alpha_{\kappa_0}) y\rangle + o(\|y\|^2)$$

(resp.

$$J_{\kappa_0} \circ \varphi^{-1}(y) - \psi \circ \varphi^{-1}(y) = \left[\left|\frac{\partial \psi \circ \varphi^{-1}}{\partial y_1}(0)\right| - \frac{\partial \psi \circ \varphi^{-1}}{\partial y_1}(0)\right] y_1$$
$$+ \langle y, C_0(\alpha_{\kappa_0} - 1) y\rangle + o(\|y\|^2)),$$

where we can assume $y_0 = 0$. Since the matrices $C_0(\alpha_{\kappa_0})$ and $C_0(\alpha_{\kappa_0} - 1)$ are positive definite, there exists a neighborhood U_{κ_0} of s_{κ_0} such that

$$\underset{s \in U_{\kappa_0} \setminus \{s_{\kappa_0}\}}{\forall} \quad J_{\kappa_0}(s) > 0$$

and

$$\underset{s \in U_{\kappa_0}}{\forall} \; J_{\kappa_0}(s) - \psi(s) \geqslant 0.$$

Consider the set $S_1 := S \setminus U_{\kappa_0}$ and let

$$E_0 := \max_{s \in S_1} |\alpha_{\kappa_0} \psi(s)| \qquad \text{and} \qquad D_0 := \min_{s \in S_1} D_{\kappa_0}(s) > 0.$$

Then we have for all $a_{\kappa_0} > E_0/D_0$ and for all $s \in S_1$ the estimate

$$a_{\kappa_0} D_{\kappa_0}(s) + \alpha_{\kappa_0} \psi(s) \geqslant D_0 a_{\kappa_0} - E_0 > 0,$$

which proves the claims in the case $\operatorname{grad} \psi(s_{\kappa_0}) \neq 0$.

To prove the claim (3) in the case $\operatorname{grad} J_{\kappa_0}(s_{\kappa_0}) = 0$, we consider second-order Taylor expansion of $J_{\kappa_0} - \psi$ at the point s_{κ_0}:

$$\begin{aligned} &J_{\kappa_0} \circ \varphi^{-1}(y) - \psi \circ \varphi^{-1}(y) \\ &\quad = \langle y - y_0, C_0(\alpha_{\kappa_0} - 1)(y - y_0)\rangle + o(\| y - y_0 \|^2) \end{aligned}$$

Since the matrix $C_0(\alpha_{\kappa_0} - 1)$ is positive definite, there exists a neighborhood U_{κ_0} of s_{κ_0} such that

$$\underset{s \in U_{\kappa_0}}{\forall} \; J_{\kappa_0}(s) - \psi(s) \geqslant 0.$$

Now define for each $b > 0$ the function

$$G_b(s) := b \prod_{\kappa=1}^{K} J_\kappa(s),$$

where $K := \# N$. The function G_b is of class C^k and has the property

$$\begin{aligned} G_b(s) &= 0 \qquad \text{if} \quad s = s_\kappa, \\ &> 0 \qquad \text{if} \quad s \neq s_\kappa. \end{aligned}$$

Next we will show that there is a real number $b > 0$ such that

$$\underset{s \in S}{\forall} \; G_b(s) \geqslant \psi(s).$$

If not, then there exist sequences $(b_j) \subset \mathbb{R}$ and $(s_{b_j}) \subset S$ such that

$$b_j \to \infty \qquad \text{and} \qquad s_{b_j} \to s_0,$$

and such that

$$\underset{j \in \mathbb{N}}{\forall} \; 1 < b_j < b_{j+1} \qquad \text{and} \qquad G_{b_j}(s_{b_j}) < \psi(s_{b_j}).$$

We claim that s_0 is different from s_κ, $\kappa = 1, 2, \ldots, K$. By claim (3) and by the remark after it, there is for each s_{κ_0} in N a neighborhood U_{κ_0} such that

$$\underset{s \in U_{\kappa_0}}{\forall} \; J_{\kappa_0}(s) - \psi(s) \geqslant 0,$$

and such that all functions J_κ with $\kappa \neq \kappa_0$ satisfy in U_{κ_0} the estimate $J_\kappa \geqslant 1$. Then we have in U_{κ_0} the estimate

$$G_1(s) - \psi(s) = \prod_{\kappa=1}^{K} J_\kappa(s) - \psi(s) \geqslant J_{\kappa_0}(s) - \psi(s) \geqslant 0.$$

Assume now that $s_0 = s_{\kappa_0}$. Then we have $s_{b_j} \in U_{\kappa_0}$ for j large enough. Consequently, we have

$$\begin{aligned} 0 &> G_{b_j}(s_{b_j}) - \psi(s_{b_j}) \\ &\geqslant G_1(s_{b_j}) - \psi(s_{b_j}) \geqslant 0, \end{aligned}$$

which is a contradiction.

Consequently, there exists a compact neighborhood U_0 of s_0 and real number α such that

$$\underset{s \in U_0}{\forall} \quad G_1(s) \geqslant \alpha > 0.$$

For j sufficiently large we have $s_{b_j} \in U_0$ and

$$b_j \alpha \geqslant \max_{s \in U_0} \psi(s);$$

then we have

$$\begin{aligned} G_{b_j}(s_{b_j}) &\geqslant b_j \alpha \geqslant \max_{s \in U_0} \psi(s) \\ &\geqslant \psi(s_{b_j}) > G_{b_j}(s_{b_j}), \end{aligned}$$

which is a contradiction.

Consequently, there exists a nonnegative number b such that

$$\underset{s \in S}{\forall} \quad P(s) := G_b(s) \geqslant \psi(s);$$

since $G_b(s) \geqslant 0$, we have also $P(s) \geqslant 0$ for all $s \in S$. It is easy to see, that P has the points of N as special zeros, which completes the proof.

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